

TWISTORS, 4-SYMMETRIC SPACES AND INTEGRABLE SYSTEMS

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1. INTRODUCTION

The theory of harmonic maps of surfaces has been greatly enriched by ideas and methods from integrable systems [4, 9, 14, 16, 18, 19]. More recently, Hélein–Romon [6, 7, 8] have applied similar ideas in their study of Hamiltonian stationary Lagrangian surfaces in 4-dimensional Hermitian symmetric spaces. The integrable system arising in this latter theory is a special case of one discussed by Terng [15] for which a geometric interpretation seemed lacking. It is the purpose of this paper to remedy that lack.

The underlying algebraic structure of the situation is a Lie algebra equipped with an automorphism of order 4. Geometrically, this means we have to do with a (locally) 4-symmetric space and the integrable system we study amounts to an equation on maps from a Riemann surface into this space. We begin by observing the 4-symmetric spaces may be viewed as submanifolds of the twistor space of an associated Riemannian symmetric space (this is further elaborated in [11]) and then our equation becomes the demand that the map into twistor space be a vertically harmonic twistor lift. When the Riemannian symmetric space is 4-dimensional, twistor lifts of conformal immersions are canonically defined and we show that vertical harmonicity amounts to holomorphicity of the mean curvature vector. In particular, we find that conformal immersions of Riemann surfaces in 4-dimensional space-forms with holomorphic mean curvature vector constitute an integrable system.

2. AN INTEGRABLE SYSTEM

Let us begin by describing an integrable system that has arisen in the general theory of Terng [15] and work of Hélein–Romon [6, 7, 8] and Khemar [10].

Our first ingredient is a Lie algebra \mathfrak{g} together with an automorphism $\tau \in \text{Aut}(\mathfrak{g})$ of order 4. We have an eigenspace decomposition

$$(1) \quad \mathfrak{g}^{\mathbb{C}} = \mathfrak{g}_0 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_{-1}$$

so that τ has eigenvalue $e^{2\pi i k/4}$ on \mathfrak{g}_k .

Now let Σ be a Riemann surface and $\alpha \in \Omega_{\Sigma}^1 \otimes \mathfrak{g}$ be a \mathfrak{g} -valued 1-form on Σ . The complex structure J^{Σ} of Σ gives a type decomposition $\alpha = \alpha^{1,0} + \alpha^{0,1}$ while (1) gives a second decomposition:

$$\alpha = \alpha_0 + \alpha_2 + \alpha_1 + \alpha_{-1}.$$

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With this in hand, our integrable system, the *second elliptic* (\mathfrak{g}, τ) -system of [15], comprises the following equations on α :

$$(2a) \quad \alpha_1^{0,1} = 0$$

$$(2b) \quad d\alpha_2^{1,0} + [\alpha_0 \wedge \alpha_2^{1,0}] = 0$$

$$(2c) \quad d\alpha + \frac{1}{2}[\alpha \wedge \alpha] = 0.$$

The equations (2) have a *zero-curvature* formulation: for $\lambda \in \mathbb{C}^\times$, define $\alpha_\lambda \in \Omega_\Sigma^1 \otimes \mathfrak{g}^\mathbb{C}$ by

$$\alpha_\lambda = \lambda^2 \alpha_2^{1,0} + \lambda \alpha_1^{1,0} + \alpha_0 + \lambda^{-1} \alpha_{-1}^{0,1} + \lambda^{-2} \alpha_2^{0,1}.$$

Then, in the presence of (2a), equations (2b) and (2c) are equivalent to the demand that $d\alpha_\lambda + \frac{1}{2}[\alpha_\lambda \wedge \alpha_\lambda] = 0$ (thus $d + \alpha_\lambda$ is a flat connection), for all $\lambda \in \mathbb{C}^\times$.

Thus the methods of integrable system theory (see, for example, [2]) apply to give generalised Weierstrass formulae, algebro-geometric solutions, spectral deformations and so on.

Our purpose in this note is to describe the geometry behind this integrable system.

3. 4-SYMMETRIC SPACES AND TWISTOR SPACES

Let $\sigma = \tau^2$ so that $\sigma^2 = 1$ and we have a symmetric decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$$

with $\mathfrak{k}^\mathbb{C} = \mathfrak{g}_0 \oplus \mathfrak{g}_2$ and $\mathfrak{p}^\mathbb{C} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_1$. Without loss of generality, we assume that $\text{ad}_{|\mathfrak{p}} : \mathfrak{k} \rightarrow \text{End}(\mathfrak{p})$ injects (any kernel is a τ -invariant ideal of \mathfrak{g} that we factor out). We then have:

$$(3a) \quad \mathfrak{g}_0 = \{\xi \in \mathfrak{g}^\mathbb{C} : [\text{ad } \xi|_{\mathfrak{p}}, \tau|_{\mathfrak{p}}] = 0\}$$

$$(3b) \quad \mathfrak{g}_2 = \{\xi \in \mathfrak{g}^\mathbb{C} : \{\text{ad } \xi|_{\mathfrak{p}}, \tau|_{\mathfrak{p}}\} = 0\}$$

where here, and below, $\{, \}$ is anti-commutator.

Now let us integrate our setup: let (G, K) be a symmetric pair corresponding to (\mathfrak{g}, σ) , that is, G is a Lie group with Lie algebra \mathfrak{g} with an involution $\sigma \in \text{Aut}(G)$ integrating our $\sigma \in \text{Aut}(\mathfrak{g})$ and $(G^\sigma)_0 \leq K \leq G^\sigma$. Thus G/K is a symmetric space and we assume henceforth that K is compact so that G/K is a Riemannian symmetric space.

Define $H \leq G$ by

$$H = \{h \in K : \text{Ad}(h) \circ \tau|_{\mathfrak{p}} = \tau|_{\mathfrak{p}} \circ \text{Ad}(h)\}$$

From (3), we deduce that H is the stabiliser of τ in K and so has Lie algebra $\mathfrak{h} = \mathfrak{g}_0 \cap \mathfrak{g}$. Thus G/H is a locally 4-symmetric space¹. By construction, G/H fibres over the Riemannian symmetric space G/K and we claim that this fibration factors through the twistor fibration of G/K .

For this, recall that the twistor space $J(N)$ of an even-dimensional Riemannian manifold is the bundle of orthogonal almost complex structures on TN :

$$J(N) = \{j \in \mathfrak{so}(TN) : j^2 = -1\}.$$

In the case at hand, we have the standard identification $T_e K G/K = \mathfrak{p}$ so that $\tau|_{\mathfrak{p}}$ is such an orthogonal² almost complex structure. Extending by equivariance, we see that the fibration $\pi : G/H \rightarrow G/K$ factors $G/H \hookrightarrow J(G/K) \rightarrow G/K$.

¹In fact, in our examples below, G will admit $\tau \in \text{Aut}(G)$ with $\tau^4 = 1$ integrating τ and H will be open in $\text{Fix}(\tau)$ so that G/H is globally 4-symmetric.

²That $\tau|_{\mathfrak{p}}$ is orthogonal for some K -invariant inner product on \mathfrak{p} is the content of [11, Theorem 21].

Remark. In [11] it is shown that, up to covers, essentially all locally 4-symmetric spaces arise as submanifolds of the twistor space of a Riemannian symmetric space in this way.

Given a Riemannian symmetric space G/K , it is natural to ask which G -orbits in $J(G/K)$ arise by the above procedure. There is a necessary condition: under the identification $T_{eK}G/K = \mathfrak{p}$, the curvature operator at eK is given by $R_{eK}(X, Y) = -\text{ad}[X, Y]_{|\mathfrak{p}}$ so that, since $\tau \in \text{Aut}(\mathfrak{g})$, we have, for $j = \tau|_{\mathfrak{p}}$,

$$(4) \quad R(jX, jY) = j \circ R(X, Y) \circ j^{-1},$$

for $X, Y \in T_{\pi(j)}G/K$. In fact, in most cases, this condition is also sufficient:

Theorem 3.1 ([11]). *Let G/K be a simply-connected Riemannian symmetric space on which G acts effectively and $j \in J(G/K)$. Then the G -orbit of j is a locally 4-symmetric space arising as above from some $\tau \in \text{Aut}(\mathfrak{g})$ of order 4 if and only if (4) holds.*

Remark. In fact, Theorem 3.1 holds in rather more generality. A problem arises only for flat spaces of the form $\mathbb{R}^k \times T^{n-k}$ where the curvature condition is vacuous but orbits by the isometry group of those j which do not respect the product structure fail to be locally 4-symmetric. See [11, Theorem 10] for more details.

4. VERTICALLY HARMONIC TWISTOR LIFTS

Let us now return to the integrable system (2): a solution α of (2) integrates, by virtue of (2c), at least locally, to give a map $g : \Sigma \rightarrow G$ with $g^{-1}dg = \alpha$ and thus a map $j = gH : \Sigma \rightarrow G/H$. Since the system is gauge-invariant, (replace g by gh for any $h : \Sigma \rightarrow H$), it is the map j that carries the geometry.

With $N = G/K$, we view $j : \Sigma \rightarrow G/H \subset J(N)$ as a map into twistor space and set $\phi = \pi \circ j : \Sigma \rightarrow N$. We may therefore view j as an orthogonal almost complex structure on $\phi^{-1}TN$. Any local frame $g : \Sigma \rightarrow G$ with $j = gH$ and $\alpha = g^{-1}dg$ gives an isomorphism $\phi^{-1}TN^{\mathbb{C}} \cong \Sigma \times \mathfrak{p}$ under which j corresponds to $\tau|_{\mathfrak{p}}$; $d\phi$ with $\alpha_1 + \alpha_{-1}$ and $d + \alpha_0 + \alpha_2$ with $\bar{\nabla}$, the Levi-Civita connection of N , pulled back by ϕ . Let D be the connection on $\phi^{-1}TN$ corresponding in this way to $d + \alpha_0$. From (3), we have that $(d + \alpha_0)\tau|_{\mathfrak{p}} = 0$ giving $Dj = 0$ and also that $\text{ad } \alpha_2$ anti-commutes with $\tau|_{\mathfrak{p}}$ whence $\bar{\nabla} - D$ anti-commutes with j . It follows at once that α_2 corresponds to $\frac{1}{2}j(\bar{\nabla}j)$ so that $D = \bar{\nabla} - \frac{1}{2}j(\bar{\nabla}j)$. We can now read off the significance of the equations (2):

First (2a) is exactly the assertion that ϕ is holomorphic with respect to j : $d\phi \circ J^{\Sigma} = j \circ d\phi$. This means that j is a *twistor lift* of ϕ . As a consequence, ϕ is a (branched) conformal immersion.

Equation (2b) amounts to the demand that $d^D j(\bar{\nabla}j)^{1,0} = 0$. We then have the complex conjugate equation $d^D j(\bar{\nabla}j)^{0,1} = 0$ and, together, these are equivalent to

$$(5a) \quad 0 = d^D j(\bar{\nabla}j)$$

$$(5b) \quad 0 = d^D * j(\bar{\nabla}j).$$

The first of these is a consequence of (2a) and (4). For this, write

$$\mathfrak{so}(\phi^{-1}TN) = \mathfrak{so}_+(\phi^{-1}TN) \oplus \mathfrak{so}_-(\phi^{-1}TN)$$

for the D -parallel, symmetric decomposition of $\mathfrak{so}(\phi^{-1}TN)$ into j -commuting and j -anti-commuting parts and let $\pi_- : \mathfrak{so}(\phi^{-1}TN) \rightarrow \mathfrak{so}_-(\phi^{-1}TN)$ be the (orthogonal) projection along $\mathfrak{so}_+(\phi^{-1}TN)$. Then $\bar{\nabla}j$ takes values in $\mathfrak{so}_-(\phi^{-1}TN)$ and, since

$[\mathfrak{so}_-(\phi^{-1}TN), \mathfrak{so}_-(\phi^{-1}TN)] \subset \mathfrak{so}_+(\phi^{-1}TN)$, $D = \pi_- \circ \bar{\nabla}$ on $\mathfrak{so}_-(\phi^{-1}TN)$. We now compute:

$$0 = d^D j(\bar{\nabla} j) = j d^D \bar{\nabla} j = j \pi_- d \bar{\nabla} j = j \pi_- [R^{\bar{\nabla}}, j] = j [R^{\bar{\nabla}}, j].$$

Thus (5a) is equivalent to $R^{\bar{\nabla}}$ commuting with j which is an immediate consequence of Theorem 3.1 along with the observation that $d\phi(T\Sigma)$ is j -stable in view of (2a).

On the other hand, equation (5b) is the vertical part of a harmonic map equation for j :

$$0 = *d^D * (j \bar{\nabla} j) = j \pi_- (*d \bar{\nabla} * \bar{\nabla} j) = -j \pi_- \bar{\nabla}^* \bar{\nabla} j.$$

This is exactly the condition that j be a harmonic section of $J(\phi^{-1}TN)$ in the sense of C.M. Wood [17, see Theorem 4.2(c)]. We say that such a twistor lift is *vertically harmonic*.

To summarise:

Theorem 4.1. *Let $j : \Sigma \rightarrow G/H \subset J(G/K)$ be a map of a Riemann surface into the twistor space of a Riemannian symmetric space G/K which factors through a locally 4-symmetric space as in section 3. Let $\phi = \pi \circ j$.*

Then j admits local frames g with $\alpha = g^{-1}dg$ solving (2) if and only if

- (1) *j is a twistor lift: $j \circ d\phi = d\phi \circ J^\Sigma$;*
- (2) *j is vertically harmonic: $[\bar{\nabla}^* \bar{\nabla} j, j] = 0$.*

Moreover, in this case, $[R^{\bar{\nabla}}, j] = 0$.

It is interesting that our integrable system is solely concerned with the geometry of j *qua* map into twistor space. The only role played by the locally 4-symmetric space G/H is to provide a (possibly empty) algebraic constraint on j and a curvature identity.

5. 4 DIMENSIONS AND HOLOMORPHIC MEAN CURVATURE VECTOR

We have seen that our theory is one about twistor lifts of conformal immersions in a symmetric space. In general, there are many such lifts but, in favourable circumstances, there are distinguished lifts and then our theory is one of the conformal immersions themselves.

In particular, suppose that $\phi : \Sigma \rightarrow N$ is a conformal immersion into an oriented 4-manifold. The twistor space of N has two components $J_\pm(N)$, each a S^2 -bundle over N , and there are unique twistor lifts $j_\pm : \Sigma \rightarrow J_\pm(N)$ given by choosing one of the two orthogonal almost complex structures on the normal bundle of ϕ . The vertical harmonicity of these twistor lifts has been studied by Hasegawa [5].

In this situation, we have a splitting $\phi^{-1}TN = T\Sigma \oplus N\Sigma$ and a corresponding decomposition

$$\bar{\nabla} = \begin{pmatrix} \nabla^\Sigma & -\Pi^t \\ \Pi & \nabla^\perp \end{pmatrix}$$

where $\nabla^\Sigma, \nabla^\perp$ are, respectively, the Levi-Civita and normal connections of Σ and $\Pi \in \Omega_\Sigma^1 \otimes \text{Hom}(T\Sigma, N\Sigma)$ is the second fundamental form of the immersion.

To explicate the vertical harmonicity of $j = j_\pm$, write $\Pi = \Pi_+ + \Pi_-$ where Π_+ commutes with j and Π_- anti-commutes. Both $j|_{T\Sigma}$ and $j|_{N\Sigma}$ are rotations through $\pi/2$ and so $\nabla^\Sigma j = \nabla^\perp j = 0$. We deduce that

$$\begin{aligned} \frac{1}{2} j \bar{\nabla} j &= \frac{1}{2} j [\Pi - \Pi^t, j] = \Pi_- - \Pi_-^t \\ D &= \nabla^\Sigma + \nabla^\perp + \Pi_+ - \Pi_+^t. \end{aligned}$$

Moreover, $[(\Pi - \Pi^t)_+ \wedge *(\Pi - \Pi^t)_-]$ must vanish as it takes values in $\mathfrak{so}(T\Sigma) \oplus \mathfrak{so}(N\Sigma)$ and anti-commutes with j and we therefore conclude that j is vertically harmonic if and only if

$$(6) \quad d^{\nabla^\Sigma, \nabla^\perp} * \Pi_- = 0.$$

In case j factors through a locally 4-symmetric space, (6) admits a simple interpretation thanks to (4) and the Codazzi equation. For this, use $j|_{N\Sigma}$ to view $N\Sigma$ as a complex line bundle and then equip that line bundle with the Koszul–Malgrange holomorphic structure whose $\bar{\partial}$ -operator is $(\nabla^\perp)^{0,1}$. We now have:

Theorem 5.1. *Let $\phi : \Sigma \rightarrow G/K$ be a conformal immersion of a Riemann surface into a 4-dimensional Riemannian symmetric space G/K and let $j : \Sigma \rightarrow G/H \subset J(G/K)$ be a twistor lift of ϕ factoring through a locally 4-symmetric space G/H as in Section 3.*

Then j is vertically harmonic if and only if the mean curvature vector $\mathbf{H} = \frac{1}{2} \text{trace } \Pi$ is a holomorphic section of $(N\Sigma, j|_{N\Sigma})$.

Proof. Let $\nabla \Pi$ denote the covariant derivative of Π with respect to the connection on $T^*\Sigma \otimes \text{Hom}(T\Sigma, N\Sigma)$ induced by $\nabla^\Sigma, \nabla^\perp$. For $X \in T\Sigma$, we have

$$(7) \quad (*d^{\nabla^\Sigma, \nabla^\perp} * \Pi)X = (R^{\bar{\nabla}}(e_i, X)e_i)^\perp + 2\nabla_X^\perp \mathbf{H}.$$

Indeed, with $e_1, e_2 = je_1$ a local frame of $T\Sigma$,

$$\begin{aligned} (*d^{\nabla^\Sigma, \nabla^\perp} * \Pi)X &= (\text{trace } \nabla \Pi)X = (\nabla_{e_i} \Pi)_{e_i} X \\ &= (\nabla_{e_i} \Pi)_X e_i \\ &= (d^{\nabla^\Sigma, \nabla^\perp} \Pi)_{e_i, X}(e_i) + (\nabla_X \Pi)_{e_i} e_i \\ &= (d^{\nabla^\Sigma, \nabla^\perp} \Pi)_{e_i, X}(e_i) + \text{trace } \nabla_X \Pi \\ &= (R^{\bar{\nabla}}(e_i, X)e_i)^\perp + 2\nabla_X^\perp \mathbf{H}, \end{aligned}$$

where we have used, in order, the symmetry of Π ; that ∇^Σ is torsion-free; the Codazzi equation and that trace is ∇^Σ -parallel.

Moreover, observe that the endomorphism $X \mapsto (R^{\bar{\nabla}}(e_i, X)e_i)^\perp$ commutes with j by virtue of (4):

$$j(R^{\bar{\nabla}}(e_i, X)e_i)^\perp = (R^{\bar{\nabla}}(je_i, jX)je_i)^\perp = (R^{\bar{\nabla}}(e_i, jX)e_i)^\perp$$

so that, applying the $\nabla^\Sigma, \nabla^\perp$ -parallel projection π_- to (7), we have:

$$*d^{\nabla^\Sigma, \nabla^\perp} * \Pi_- = \pi_-(*d^{\nabla^\Sigma, \nabla^\perp} * \Pi) = 2\pi_-(\nabla^\perp \mathbf{H}).$$

We conclude that j is vertically harmonic if and only if $\nabla^\perp \mathbf{H}$ commutes with j or, equivalently, \mathbf{H} is a holomorphic section of $(N\Sigma, j|_{N\Sigma})$. \square

5.1. Examples. Let us enumerate the 4-symmetric spaces that fibre over a simply-connected 4-dimensional Riemannian symmetric space N . There are three cases:

5.1.1. N has constant sectional curvatures. Here both $J_+(N)$ and $J_-(N)$ are themselves globally 4-symmetric so there is no algebraic constraint on j to take into account. We therefore recover a theorem of Hasegawa:

Theorem 5.2 ([5]). *A conformal immersion $\phi : \Sigma \rightarrow N$ of a Riemann surface into a simply connected 4-manifold of constant sectional curvature has vertically harmonic twistor lift j_\pm if and only if its mean curvature vector is holomorphic in $(N\Sigma, (j_\pm)|_{N\Sigma})$.*

It is interesting that these surfaces already solve an integrable system in conformal geometry: they are constrained Willmore surfaces (see [3] and also Bohle [1] for the case $N = \mathbb{R}^4$).

5.1.2. *N has constant holomorphic sectional curvatures.* For any Kähler 4-manifold N , fix the orientation so that the ambient complex structure J^N is a section of $J_+(N)$. If N has constant holomorphic sectional curvatures, $J_-(N) = \{j \in J(N) : [j, J^N] = 0\}$ is again 4-symmetric (for example, if $N = \mathbb{C}P^2$, $J_-(N)$ is the full flag manifold of $SU(3)$). Again there is no algebraic constraint to take into account and we conclude:

Theorem 5.3. *A conformal immersion $\phi : \Sigma \rightarrow N$ of a Riemann surface into a simply connected 4-manifold of constant holomorphic sectional curvature has vertically harmonic twistor lift j_- if and only if its mean curvature vector is holomorphic in $(N\Sigma, (j_-)_{|N\Sigma})$.*

5.1.3. *N is Hermitian symmetric.* Again we fix orientations so that the ambient complex structure J^N is a section of $J_+(N)$ and now contemplate the subbundle $Z = \{j \in J_+(N) : \{j, J^N\} = 0\}$ of almost complex structures that anti-commute with J^N . This is a circle bundle over N (it is the unit circle bundle in the canonical bundle of N) and is 4-symmetric.

In this case, we do have an algebraic constraint of the twistor lift j_+ . Indeed, let ω^N be the Kähler form of N . We then have:

Lemma 5.4. *j_+ takes values in Z if and only if ϕ is Lagrangian: $\phi^*\omega^N = 0$.*

Proof. First, if ϕ is Lagrangian, J^N restricts to isometries $T\Sigma \rightarrow N\Sigma$ and $N\Sigma \rightarrow T\Sigma$ whence $J^N j_+ = \pm j_+ J^N$ and now, since j_+ takes values in $J_+(N)$, we must have $J^N j_+ = -j_+ J^N$. Thus j_+ is Z -valued.

Conversely, if $\{j, J^N\} = 0$ then one readily computes that, for $X \in T\Sigma$,

$$\phi^*\omega^N(X, J^\Sigma X) = -\phi^*\omega^N(X, J^\Sigma X)$$

so that ϕ is Lagrangian. □

Now recall the *Maslov form* of a Lagrangian immersion ϕ which is given³ by $\beta = \iota_{\mathbf{H}}\omega^N \in \Omega_\Sigma^1$. We have:

Lemma 5.5. *Let $\phi : \Sigma \rightarrow N$ be Lagrangian with twistor lift $j = j_+ : \Sigma \rightarrow Z$. Then*

$$(8) \quad \Pi_- = \beta J^N|_{T\Sigma}.$$

Proof. First note that since N is Kähler, $\overline{\nabla} J^N = 0$ whence

$$(9) \quad \Pi \circ J^N = J^N \circ \Pi^t$$

$$(10) \quad \nabla J^N = 0$$

where, as before, ∇ is the connection on $\mathfrak{so}(TN)$ induced by ∇^Σ and ∇^\perp . Taking the j -anti-commuting part of (9) yields

$$(11) \quad \Pi_- \circ J^N = J^N \circ (\Pi_-)^t.$$

By definition Π_- takes values in the subbundle of $\text{Hom}(T\Sigma, N\Sigma)$ consisting of homomorphisms that anti-commute with j . In the present situation, this subbundle is spanned by $J^N|_{T\Sigma}$ and $jJ^N|_{T\Sigma}$ and we easily see from (11) that Π points along $J^N|_{T\Sigma}$ and so is of the form $\beta J^N|_{T\Sigma}$ for some $\beta \in \Omega_\Sigma^1$.

It remains to identify β with the Maslov form. For this let $X, e \in T_p\Sigma$ with e of unit length. Then, using $\Pi_- = \frac{1}{2}(\Pi + j\Pi j)$, the skew-symmetry of j and J^N as well as the fact that these anti-commute, we have

$$\beta_X = (\Pi_- X, J^N e) = \frac{1}{2}((\Pi_X, J^N e) + (\Pi_X j e, J^N j e)).$$

³Here we have dropped a customary factor of $1/\pi$.

On the other hand, for any $Y \in T_p \Sigma$, the symmetry of \mathbb{I} along with (9) yields

$$(\mathbb{I}_X Y, J^N Y) = (\mathbb{I}_Y X, J^N Y) = (X, \mathbb{I}_Y^t J^N Y) = (X, J^N \mathbb{I}_Y Y)$$

so that we conclude that

$$\beta_X = (X, J^N \mathbf{H})$$

and we are done. \square

With this in hand, the vertical harmonicity of j_+ is easy to understand: in view of (10), we have

$$d^{\nabla^\Sigma, \nabla^\perp} * \mathbb{I}_- = (d * \beta) J^N|_{T\Sigma}$$

so that j_+ is vertically harmonic if and only if the Maslov form is co-closed which, in turn, is precisely the condition that ϕ be Hamiltonian stationary [12]. To summarise:

Theorem 5.6. *Let $\phi : \Sigma \rightarrow N$ be a conformal immersion of a Riemann surface into a simply-connected 4-dimensional Hermitian symmetric space with twistor lift $j_+ : \Sigma \rightarrow J_+(N)$.*

Then j_+ takes values in $Z = \{j \in J_+(N) : \{j, J^N\} = 0\}$ if and only if ϕ is Lagrangian and then is vertically harmonic if and only if ϕ is Hamiltonian stationary.

This provides a conceptual explanation for the integrable system appearing in the work of Hélein–Romon [6, 7, 8].

6. PROSPECTS

We have seen that the integrable system (2) gives a satisfying geometric theory for conformal immersions of Riemann surfaces in 4-dimensional symmetric spaces. It is natural to ask whether there are similar theories in case Σ has either higher codimension in a symmetric space or, indeed, higher dimension. We briefly examine these possibilities.

6.1. Higher codimension. When $\dim N > 4$, there are, in general, no canonically defined twistor lifts. However, there are still interesting examples. Here is one such: take $N = \mathbb{R}^8$, identified with the octonions and contemplate the unit sphere $S^6 \subset \text{Im } \mathbb{O}$. Left multiplication by $q \in S^6$ is an orthogonal complex structure on \mathbb{O} so that we can view the trivial S^6 -bundle over N as a submanifold of $J(N)$ which is once more a 4-symmetric space. Moreover, if $\phi : \Sigma \rightarrow N$ is a conformal immersion of a Riemann surface, we get a map $\Sigma \rightarrow S^6$ by $p \mapsto q_1 \bar{q}_2$, for q_1, q_2 an oriented orthonormal basis of $d\phi(T_p \Sigma)$, and thus a map $j : \Sigma \rightarrow J(N)$. It is easy to see that j is a twistor lift of ϕ and our theory applies to show that the ϕ with vertically harmonic j (the ρ -harmonic surfaces of [10]) constitute an integrable system. Thus we recover the results of Khemar [10].

However, the geometric interpretation of ρ -harmonic surfaces is not as straightforward as in codimension 2: both $\nabla^\perp j$ and $[\mathbb{I}_+^t \wedge \mathbb{I}_-^{1,0}]$ contribute non-trivial terms to (5). We may return to this elsewhere.

6.2. Higher dimension. It is well-known that the integrable systems approach to harmonic maps of Riemann surfaces extends essentially without adjustment to treat pluriharmonic maps of Kähler manifolds: see, for example, [4, 13]. The same is true in the present case.

For this, with \mathfrak{g}, τ as in Section 2, let Σ be a complex manifold and $\alpha \in \Omega_{\Sigma}^1 \otimes \mathfrak{g}$. In this context, the integrable system (2) can be reformulated as:

$$(12a) \quad \alpha_1^{0,1} = 0$$

$$(12b) \quad \bar{\partial}\alpha_2^{1,0} + [\alpha_0^{0,1} \wedge \alpha_2^{1,0}] = 0$$

$$(12c) \quad d\alpha + \frac{1}{2}[\alpha \wedge \alpha] = 0.$$

This again has a zero curvature representation so long as $\mathfrak{h} = \mathfrak{g}_0 \cap \mathfrak{g}$ is compact: indeed, with α_{λ} defined as before, the flatness of $d + \alpha_{\lambda}$ amounts to (12) along with $[\alpha_2^{1,0} \wedge \alpha_2^{1,0}] = 0$. However, we can argue as in [13] starting with $(\partial\bar{\partial} + \bar{\partial}\partial)\alpha_2^{1,0} = 0$ and using (12) to conclude that

$$[[\alpha_2^{1,0} \wedge \alpha_2^{1,0}] \wedge \alpha_2^{0,1}] = 0.$$

The vanishing of $[\alpha_2^{1,0} \wedge \alpha_2^{1,0}]$ now follows by contracting this against $\alpha_2^{0,1}$ and using the definiteness of the Killing form on \mathfrak{h} .

We can now repeat the analysis of Section 4, *mutatis mutandis*, to conclude:

Theorem 6.1. *Let $j : \Sigma \rightarrow G/H \subset J(G/K)$ be a map of a complex manifold into the twistor space of a Riemannian symmetric space G/K which factors through a locally 4-symmetric space as in section 3. Let $\phi = \pi \circ j$.*

Then j admits local frames g with $\alpha = g^{-1}dg$ solving (12) if and only if

- (1) *j is a twistor lift: $j \circ d\phi = d\phi \circ J^{\Sigma}$;*
- (2) *j is vertically pluriharmonic: $[\bar{\partial}\bar{\nabla}(\bar{\nabla}j)^{1,0}, j] = 0$.*

Moreover, in this case, $[R^{\bar{\nabla}}, j]^{1,1} = 0$.

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